Renewable Energy Department

Lecture of Differential equations

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Mathematics – (III)

First- order and first degree D.E, initial value problem, variable separable, homogeneous and non homogeneous D.E, Exact equation, linear equation, integrating factor, Bernoulli equation, orthogonal trajectories, application physics Reduction the order of the second order differential equation to first order equation, n – order differential equation, linear non homogeneous D.E of order n, Wronskian determinant, An existence and unique theorem, second order linear homogeneous D.E with constant coefficients, Laplace transforms, inverse Laplace transforms, solution of initial value problem by Laplace transforms, definitions of partial and Fourier series.

Reference.

1- Differential Equations 1,MATB44H3F,Version September 15,2011-1049.
2- Ordinary Differential Equations a first course ,Freed Brauer and John A.Nothel.1973.

Differential equation:

A differential equation (D.E) is an equation involving a function and its derivatives.

Example:

$1-\frac{dy}{dx}+ycosx=sinx$

$2-y'\ cosx-3\quad is\ no\ differential\ equation$

Ordinary differential equation:

A differential equation (D.E) is called ordinary differential equation if not contain partial derivatives.

Example:

$1-\frac{\mathrm{dy}}{\mathrm{dx}}+\mathrm{yx}=4\mathrm{x}-1$ is ordinary differential equation

$2-x\frac{dy}{dx}+y\frac{dz}{dy}=z$ is partial differential equation

is not ordinary differential equation is partial derivatives

Order of ordinary differential equation:

The order of ordinary differential equation is the highest order derivative occurring.

Example:

$1-\frac{d^2y}{dx^2}+3y-2x=0$

second order of ordinary differential equation

$$2-\frac{d^2y}{dx^2}+3y'''-2x=\frac{d^4y}{dx^4}$$

frourth order of ordinary differential equation

Degree of ordinary differential equation:

The degree of ordinary differential equation is the highest power or power for highest derivative occurring.

Example:

$$1-\left(\frac{d^2y}{dx^2}\right)^2+3y'-2x=0$$

second degree of ordinary differential equation

$$2-\left(\frac{d^2y}{dx^2}\right)^2+3y''''-2x=sinx$$

first degree of ordinary differential equation

Solution of differential equation of order n:

The Solution of differential equation of order n consists of a function defined and n times differentiable on a domain D having the property that the functional equation obtained by substituting the function and its n derivatives into the differential equation holds for every point in D.

Example:

If the function $y=sinx$ is a solution of

$$y''+y=0$$

Solution.

$y = sinx$

$y' = cosx$

$y'' = -sinx$

$y'' + y = 0$

$-sinx + sinx = 0$

$y = sinx$ is a solution of $y'' + y = 0$

Exercise:

1-If the function $y = e^{-2x}$ is a solution of

$y''' - 4y'' - 4y' + 16y = 0$

2- If the function $y = Ln\, x^3$ is a solution of

$y''' - 4y'' + 16y = 3$

First- order differential equation:

1-Variable separable differential equation

2-Homogeneous differential equation

3- Non homogeneous differential equation

4-Exact differential equation

5-Integrating factor differential equation

6-linear equation differential equation

7-Bernoulli differential equation

1-Variable separable differential equation:

A first order has the form $F(x, y, y') = 0$

Such that

$$F(x, y, y') = 0$$

$$y' = f(x, y)$$

$$\frac{dy}{dx} = g(x).h(y) \ [\div h(y)] \ \ [* d(x)]$$

$$\frac{dy}{h(y)} = g(x)dx \quad h(y) \neq 0$$

$$\int \frac{dy}{h(y)} = \int g(x)dx$$

Example:

Find the general solution for differential equation.

$y' = e^{x-y}$

Solution:

$y' = e^{x-y}$

$\frac{dy}{dx} = e^x \ e^{-y} \qquad [* dx]$

$\frac{dy}{dx}dx = e^x \ e^{-y}dx$

$dy = e^x \ e^{-y} \ dx \qquad [\div e^{-y}]$

$\frac{dy}{e^{-y}} = e^x \ dx$

$e^y dy = e^x \ dx$

$$\int e^{y} dy = \int e^{x} \, dx$$

$e^{y} = e^{x} + c \qquad [* \, Ln] \qquad [\, e^{lnx} = x, \qquad lne^{x} = x]$

$Ln \; e^{y} = Ln \;\; (e^{x} + c)$

$y = \; Ln \;\; (e^{x} + c)$

$y = x + Lnc$

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<u>Initial value problem:</u>

The initial value problem is a condition with differential equation to get the value of C

<u>Example:</u>

Find the general solution for differential equation.

$y' = e^{x-y} \quad , \qquad y(-1) = 0 \;, \;\; y(x) = y$

<u>Solution:</u>

By the above example the solution is $y = x + Lnc$

the initial value problem for $y' = e^{x-y}$

$y = x + Lnc \qquad *$

$0 = -1 + Lnc$

$1 = Lnc$

$y = x + 1$

Example:

Find the general solution for differential equation.

$3x^2y^2dx + y^2dx + dy = 0, \qquad y(2) = 1.$

Solution:

$3x^2y^2dx + y^2dx + dy = 0$

$3x^2y^2dx + y^2dx + dy = 0 \quad \div y^2$

$$3x^2dx + dx + \frac{dy}{y^2} = 0$$

$$\frac{dy}{y^2} = -3x^2dx - dx$$

$$\int \frac{dy}{y^2} = \int -3x^2dx - \int dx$$

$$\int y^{-2}\,dy = \int -3x^2dx - \int dx$$

$$\frac{y^{-2+1}}{-2+1} = \frac{-3x^{2+1}}{2+1} - x + c$$

$$\frac{y^{-1}}{-1} = -x^3 - x + c$$

$$\frac{-1}{y} = -x^3 - x + c \qquad \left(\frac{-2}{3} = \frac{2}{-3}\right),\ y(2) = 1.$$

$$\frac{-1}{1} = -2^3 - 2 + c$$

$$-1 = -8 - 2 + c$$

$$9 = c$$

$$\frac{-1}{y} = -\mathrm{x}^3 - \mathrm{x} + 9$$

$$\mathrm{y} = \frac{-1}{-\mathrm{x}^3 - \mathrm{x} + 9}$$

Example:

Find the general solution for differential equation.

$$dxlnx + dy = 0$$

Solution:

$$dxlnx + dy = 0$$

$$dy = -\ln x\, dx$$

$$\int \quad dy = -\int \ln x\, dx$$

$$\mathrm{y} = -(\mathrm{xlnx} - \mathrm{x}) + \mathrm{c}$$

$$\mathrm{y} = -\mathrm{xlnx} + \mathrm{x} + \mathrm{c}$$

Exercise:

$1 - (1 + x^2)y' = 1 + y^2$

$2 - (xy^2 + x)dx + (yx^2 + y)dy = 0$

$3 - y'siny = sin^2x$

$4 - 2e^{3x}siny\ \ dx + e^x cscydy = 0\ , \quad y(2) = 1$

$5 - xe^y dy + \dfrac{x^2 + 1}{y} dx = 0$

Homogeneous differential equation:

A function $F(x, y)$ is called homogeneous differential equation of degree n if

$F(\lambda x, \lambda y) = \lambda\, F(x, y)$

Method -1-

$$y' = \frac{y + x}{x}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y + \lambda x}{\lambda x} = \frac{\lambda(y + x)}{\lambda x} = \frac{y + x}{x} = F(x, y)$$

Method of solution-2-

$$y = vx \rightarrow v = \frac{y}{x}$$

$$y' = v + x\frac{dv}{dx}$$

Example:

Find the general solution for differential equation.

$y' = \frac{y + x}{x}$

Solution:

$$y' = \frac{y + x}{x}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y + \lambda x}{\lambda x} = \frac{\lambda(y + x)}{\lambda x} = \frac{y + x}{x} = F(x, y)$$

$$y' = \frac{y + x}{x}$$

$$v + x\frac{dv}{dx} = \frac{y + x}{x}$$

$$v + x\frac{dv}{dx} = \frac{\frac{y}{x} + \frac{x}{x}}{\frac{x}{x}}$$

$$v + x\frac{dv}{dx} = \frac{\frac{y}{x} + 1}{1}$$

$$v + x\frac{dv}{dx} = \frac{y}{x} + 1 \rightarrow v = \frac{y}{x}$$

$$v + x\frac{dv}{dx} = v + 1 \quad \rightarrow (\boldsymbol{v.s})$$

$$x\frac{dv}{dx} = 1 \quad \div x$$

$$\frac{dv}{dx} = \frac{1}{x} \quad * dx$$

$$dv = \frac{dx}{x}$$

$$\int dv = \int \frac{dx}{x}$$

$$v = Ln|x| + c$$

$$\frac{y}{x} = Ln|x| + c$$

$$y = \boldsymbol{x}Ln|x| + xc$$

Example:

Find the general solution for differential equation.

$$y' = \frac{y}{x + \sqrt{xy}}$$

Solution:

$$y' = \frac{y}{x + \sqrt{xy}}$$

$$F(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x + \sqrt{\lambda x \, \lambda y}} = \frac{\lambda y}{\lambda x + \sqrt{\lambda^2 x \, y}} = \frac{\lambda y}{\lambda x + \lambda xy}$$

$$= \frac{\lambda y}{\lambda(x + xy)} = \frac{y}{x + xy} \neq \frac{y}{x + \sqrt{xy}} = F(x, y)$$

$$y' = \frac{y}{x + \sqrt{xy}}$$

$$v + x\frac{dv}{dx} = \frac{y}{x + \sqrt{xy}}$$

$$v + x\frac{dv}{dx} = \frac{\frac{y}{x}}{\frac{x}{x} + \sqrt{\frac{x\,y}{x^2}}}$$

$$v + x\frac{dv}{dx} = \frac{\frac{y}{x}}{1 + \sqrt{\frac{y}{x}}}$$

$$v + x\frac{dv}{dx} = \frac{v}{1 + \sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{v}{1 + \sqrt{v}} - v$$

$$x\frac{dv}{dx} = \frac{v - v - v\sqrt{v}}{1 + \sqrt{v}}$$

$$x\frac{dv}{dx} = \frac{-v\sqrt{v}}{1 + \sqrt{v}} \quad \div dv$$

$$\frac{x}{dx} = \frac{-v\sqrt{v}}{1+\sqrt{v}} \quad \frac{1}{dv}$$

$$\frac{-v\sqrt{v}}{1+\sqrt{v}} \quad \frac{1}{dv} = \frac{x}{dx}$$

$$\frac{dv}{\frac{-v\sqrt{v}}{1+\sqrt{v}}} = \frac{dx}{x}$$

$$dv \quad \frac{1+\sqrt{v}}{-v\ \sqrt{v}} = \frac{dx}{x}$$

$$dv \quad \frac{1+\sqrt{v}}{-v\ v^{\frac{1}{2}}} = \frac{dx}{x}$$

$$dv \ \frac{1+\sqrt{v}}{-\ v^{\frac{3}{2}}} = \frac{dx}{x}$$

$$\int \quad dv \ \frac{1+\sqrt{v}}{-\ v^{\frac{3}{2}}} = \int \quad \frac{dx}{x}$$

$$-\int \frac{1}{v^{\frac{3}{2}}} dv - \int \quad \frac{\sqrt{v}}{v^{\frac{3}{2}}} dv = \int \quad \frac{dx}{x}$$

$$-\int \frac{1}{v^{\frac{3}{2}}} dv - \int \boldsymbol{v^{\frac{1}{2}}}\ v^{\frac{-3}{2}} dv = \int \quad \frac{dx}{x}$$

$$-\int v^{\frac{-3}{2}}\ dv \ \ -\int \boldsymbol{v^{-1}}\, dv = \int \quad \frac{dx}{x}$$

$$\frac{-v^{\frac{-1}{2}}}{\frac{-1}{2}} - \int \frac{\mathbf{1}}{\boldsymbol{v}} \boldsymbol{dv} = Ln|x| + c$$

$$\frac{2}{\sqrt{v}} - \boldsymbol{Ln}\ |\boldsymbol{v}| = Ln|x| + c$$

$$\frac{2}{\sqrt{\frac{y}{x}}} - \boldsymbol{Ln}\ \left|\frac{y}{x}\right| = Lnx + c$$

<u>Example:</u>

Find the general solution for differential equation.

$$\left(xsin\frac{y}{x} - ycos\frac{y}{x}\right) dx + xcos\frac{y}{x} dy = 0$$

<u>Solution:</u>

$$\left(xsin\frac{y}{x} - ycos\frac{y}{x}\right) dx + xcos\frac{y}{x} dy = 0$$

$$\frac{dy}{dx} = \frac{-xsin\frac{y}{x} + ycos\frac{y}{x}}{xcos\frac{y}{x}}$$

$$v + x\frac{dv}{dx} = \frac{-sinv + vcosv}{cosv}$$

$$x\frac{dv}{dx} = \frac{-sinv + vcosv}{cosv} - v$$

$$x\frac{dv}{dx} = \frac{-sinv + vcosv - vcocv}{cosv}$$

$$x\frac{dv}{dx} = \frac{-sinv}{cosv}$$

$$\frac{x}{dx} = \frac{-sinv}{cosv\ dv}$$

$$\frac{-dx}{x} = \frac{cosvdv}{sinv}$$

$$\int \quad \frac{-dx}{x} = \int \quad \frac{cosv\ dv}{sinv}$$

$$-Ln|x| + c = Ln\ |sinv|$$

$$-Ln|x| + c = Ln\ \left|sin\frac{y}{x}\right|$$

$$Ln|x|^{-1} + c = Ln\ \left|sin\frac{y}{x}\right|\ * e^{x}$$

$$e^{x}(Ln|x|^{-1} + c) = e^{x}Ln\ \left|sin\frac{y}{x}\right|$$

$$|x|^{-1} + e^{x}c =\ \left|sin\frac{y}{x}\right|$$

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Exercise:

Find the general solution for differential equation.

$$1 - y' = \frac{y}{\sqrt{xy}}$$

$$2 - 2x^{2}\frac{dy}{dx} = x^{2} + y^{2}$$

$$3-2(x^2+y^2)dx-xydy=0$$

$$4-y'=\frac{2y^4+y^4}{xy^3}$$

$$5-y'=\frac{y-x}{y+x}$$

$$6-(x-ylny+ylnx)dx+x(lny-lnx)dy=0$$

$$7-\left(xe^{\frac{y}{x}}+y\right)dx-xdy=0$$

<u>3- Non Homogeneous differential equation:</u>

<u>Example:</u>

Find the general solution for differential equation.

$$y'=\frac{x-y-3}{x+y+1}$$

Solution:

$$y'=\frac{x-y-3}{x+y+1}$$

$$F(\lambda x,\lambda y)=\frac{\lambda x-\lambda y-3}{\lambda x+\lambda y+1}\neq F(x,y)$$

Non Homogeneous

$$y'=\frac{x-y-3}{x+y+1}$$

$$x-y-3=0 \quad \ldots..(1)$$

$$\underline{x+y+1=0 \quad \ldots\ldots(2)}$$

$2x - 2 = 0 \to x = 1$ $\quad\quad$ $.\, y = -2$

Let $\quad X = x - x_0 \quad\quad Y = y - y_0$

Then

$X = x - 1 \quad\quad Y = y + 2$

$x = X + 1 \quad\quad y = Y - 2$

$dx = dX \quad\quad d\,y = dY$

$$y' = \frac{x - y - 3}{x + y + 1}$$

$$\frac{dY}{dX} = \frac{X + 1 - (Y - 2) - 3}{X + 1 + Y - 2 + 1} = \frac{X - Y}{X + Y}$$

$$\frac{dY}{dX} = \frac{X - Y}{X + Y} \quad \to \mathit{Homogenuous}$$

$$v + X\frac{dv}{dx} = \frac{X - Y}{X + Y}$$

$$v + X\frac{dv}{dX} = \frac{\frac{X}{X} - \frac{Y}{X}}{\frac{X}{X} + \frac{Y}{X}}$$

$$v + X\frac{dv}{dX} = \frac{1 - \frac{Y}{X}}{1 + \frac{Y}{X}}$$

$$v + X\frac{dv}{dX} = \frac{1 - v}{1 + v}$$

$$\int \frac{v+1}{v^2+2v-1} dv = -\int \quad \frac{dX}{X}$$

$$\frac{1}{2}\int 2\frac{v+1}{v^2+2v-1} dv = -\int \quad \frac{dX}{X}$$

$$\frac{1}{2} Ln|v^2+2v-1| = -Ln|X| + c$$

$$\frac{1}{2} Ln\left|\left(\frac{y+2}{x-1}\right)^2 + 2\frac{y+2}{x-1} - 1\right| = -Ln|x-1| + c$$

Example:

Find the general solution for differential equation.

$$y' = \frac{3x-y-1}{x-y+3}$$

Solution:

$$y' = \frac{3x-y-1}{x-y+3}$$

$$F(\lambda x, \lambda y) = \frac{\lambda 3x - \lambda y - 1}{\lambda x - \lambda y + 3} \neq F(x,y)$$

Non Homogeneous

$$y' = \frac{3x-y-1}{x-y+3}$$

$3x - y - 1 = 0 \quad ...(1)$

$\underline{x - y + 3 = 0 \quad ... \quad (2)}$

$$3x - y - 1 = 0 \quad \dots (1)$$

$$\underline{-x + y - 3 = 0 \quad \dots \quad (2)}$$

$$2x - 4 = 0 \quad \rightarrow \quad x = 2 \quad . y = 5$$

Let $X = x - x_0$ $\quad Y = y - y_0$

Then

$$X = x - 2 \qquad Y = y - 5$$

$$x = X + 2 \qquad y = Y + 5$$

$$dx = dX \qquad d\,y = dY$$

$$y' = \frac{3x - y - 1}{x - y + 3}$$

$$\frac{dY}{dX} = \frac{3(X+2) - (Y+5) - 1}{(X+2) - (Y+5) + 3} = \frac{3X - Y}{X - Y}$$

$$\frac{dY}{dX} = \frac{3X - Y}{X - Y} \quad \rightarrow Homogenuous$$

$$v + X\frac{dv}{dX} = \frac{3X - Y}{X - Y}$$

$$v + X\frac{dv}{dX} = \frac{\frac{3X}{X} - \frac{Y}{X}}{\frac{X}{X} - \frac{Y}{X}}$$

$$v + X\frac{dv}{dX} = \frac{3 - \frac{Y}{X}}{1 - \frac{Y}{X}}$$

$$v + X\frac{dv}{dX} = \frac{3 - v}{1 - v}$$

$$X\frac{dv}{dX} = \frac{3 - v}{1 - v} - \boldsymbol{v}$$

$$X\frac{dv}{dX} = \frac{3 - v - v + v^2}{1 - v}$$

$$X\frac{dv}{dX} = \frac{3 - 2v + v^2}{1 - v}$$

$$\frac{x}{dX} = \frac{3 - 2v + v^2}{1 - v}\frac{1}{dv}$$

$$\frac{1 - v}{3 - 2v + v^2}dv = \frac{dX}{X}$$

$$\int \quad \frac{1 - v}{3 - 2v + v^2}dv = \int \quad \frac{dX}{X}$$

$$\frac{1}{-2}\int -2\frac{1 - v}{3 - 2v + v^2}dv = \int \quad \frac{dX}{X}$$

$$\frac{1}{-2}Ln|3 - 2v + v^2| = Ln|X| + c$$

$$\frac{1}{-2}Ln\left|3 - 2\frac{y - 5}{x - 2} + \left(\frac{y - 5}{x - 2}\right)^2\right| = Ln|x - 2 \ | + c$$

Example:

Find the general solution for differential equation.

$$y' = \frac{2x + 3y - 10}{2x + 3y + 5}$$

Solution:

$$y' = \frac{2x + 3y - 10}{2x + 3y + 5}$$

$$F(\lambda x, \lambda y) = \frac{\lambda 2x + 3\lambda y - 10}{\lambda 2x + 3\lambda y + 5} \neq F(x, y)$$

Non Homogeneous

$$y' = \frac{2x + 3y - 10}{2x + 3y + 5}$$

$2x + 3y - 10 \quad \ldots (1)$

$\underline{2x + 3y + 5 \quad \ldots \quad (2)}$

This is two parallel line

$w = 2x + 3y$

$\frac{dw}{dX} = 2 + 3\frac{dy}{dx} \quad *$

$\frac{dy}{dx} = \frac{w - 10}{w + 5}$

$\frac{dw}{dX} = 2 + 3\left(\frac{w - 10}{w + 5}\right)$

$\frac{dw}{dX} = 2 + \frac{3w - 30}{w + 5}$

$\frac{dw}{dX} = \frac{2w + 10 + 3w - 30}{w + 5}$

$\frac{dw}{dX} = \frac{5w - 20}{w + 5}$

$$\frac{dw}{dX} = \frac{5(w-4)}{w+5}$$

$$\int \frac{w+5}{w-4} dw = 5 \int dx$$

$$\int \frac{(w+5-9)+9}{w-4} dw = 5 \int dx$$

$$\int \frac{w-4}{w-4} dw + \int \frac{9}{w-4} dw = 5 \int dx$$

$$\int \quad dw + 9 \int \frac{1}{w-4} dw = 5 \int dx$$

$$w + 9Ln|w-4| = 5x + c$$

$$2x + 3y + 9Ln|2x + 3y - 4| = 5x + c$$

Example:

Find the general solution for differential equation.

$$y' = \frac{6x + 2y + 1}{2x - y + 2}$$

Solution:

$$y' = \frac{6x + 2y + 1}{2x - y + 2}$$

$$F(\lambda x, \lambda y) = \frac{6\lambda x + 2\lambda y + 1}{2\lambda x - \lambda y + 2} \neq F(x, y)$$

Non Homogeneous

$$y' = \frac{6x + 2y + 1}{2x - y + 2}$$

$6x + 2y + 1 = 0 \quad \ldots\ldots(1)$

$\underline{2x - y + 2 = 0 \quad \ldots\ldots(2) \quad * 2}$

$6x + 2y + 1 = 0 \quad \ldots\ldots(1)$

$\underline{4x - 2y + 4 = 0 \quad \ldots\ldots(2)}$

$10x + 5 = 0 \rightarrow x = -\frac{1}{2} \qquad .\, y = +1$

Let $\quad X = x - x_0 \qquad\qquad Y = y - y_0$

Then

$X = x + \frac{1}{2} \qquad Y = y - 1$

$x = X - \frac{1}{2} \qquad y = Y + 1$

$dx = dX \qquad d\,y = dY$

$$y' = \frac{6x + 2y + 1}{2x - y + 2}$$

$$\frac{dY}{dX} = \frac{6\left(X - \frac{1}{2}\right) + 2(Y + 1) + 1}{2\left(X - \frac{1}{2}\right) - (Y + 1) + 2} = \frac{6X + 2Y}{2X - Y}$$

$$\frac{dY}{dX} = \frac{6X + 2Y}{2X - Y} \quad \rightarrow Homogenuous$$

$$v + X\frac{dv}{dx} = \frac{6X + 2Y}{2X - Y}$$

$$v + X\frac{dv}{dX} = \frac{\frac{6X}{X} + \frac{2Y}{X}}{\frac{2X}{X} - \frac{Y}{X}}$$

$$v + X\frac{dv}{dX} = \frac{6 + 2\frac{Y}{X}}{2 - \frac{Y}{X}}$$

$$v + X\frac{dv}{dX} = \frac{6 + 2v}{2 - v}$$

$$X\frac{dv}{dX} = \frac{6 + 2v}{2 - v} - v$$

$$X\frac{dv}{dX} = \frac{6 + 2v - 2v + v^2}{2 - v}$$

$$X\frac{dv}{dX} = \frac{6 + v^2}{2 - v}$$

$$\int \frac{2 - v}{v^2 + 6} dv = \int \quad \frac{dX}{X}$$

$$\int \frac{2}{v^2 + 6} dv - \frac{1}{2}\int \frac{2v}{v^2 + 6} dv = \int \quad \frac{dX}{X}$$

$$\int \frac{2}{v^2 + 6} dv - \frac{1}{2} Ln|v^2 + 6| = Ln|X| + c$$

$$\frac{v^2}{2} + 3$$

$$2 \,\big|\, v^2 + 6$$

$$\int \frac{2}{v^2+6} = \int p(x) + \frac{r(x)}{g(x)}$$

Exercise:

Find the general solution for differential equation.

$$1 - y' = \frac{x+2y-1}{2x-3y+6}$$

$$2 - y' = \frac{2x+3y-10}{2x+3y+5}$$

$$3 - (x^2+y^2-2x-4y+5)y' = xy - 2x - y + 2$$

4-Exact differential equation:

M(x,y) dx+N(x,y) dy=0

$$M_y = \emptyset_x$$

$$N_x = \emptyset_y$$

$$M_y = N_x$$

Method of solution

Case one: The method of M

$$1 - \emptyset(x,y) = \int M(x,y)dx$$

$$\emptyset(x,y) = M^* + g(y)$$

$$N(x,y) = \frac{d}{dy}M^{*} + g'(y)$$

$$g(y) = \int (\emptyset_y - \frac{d}{dy}M^{*})dy$$

Case one: The method of N

$$2 - \emptyset(x,y) = \int N(x,y)dy$$

$$\emptyset(x,y) = N^{*} + h(x)$$

$$M(x,y) = \frac{d}{dx}N^{*} + h'(x)$$

$$h(x) = \int (\emptyset_x - \frac{d}{dy}N^{*})dx$$

Example:

Find the general solution for differential equation.

$(\boldsymbol{3y + e^{x}})\boldsymbol{dx} + (\boldsymbol{3x + cosy})\boldsymbol{dy} = \boldsymbol{0}$

Solution:

$(3y + e^{x})dx + (3x + cosy)dy = 0$

M(x,y) dx+ N(x,y) dy = 0

$\mathrm{M(x,y)} = 3y + e^{x} \rightarrow M_y = 3$

$\mathrm{N(x,y)} = 3x + cosy \rightarrow N_x = 3$

$M_y = N_x$ **Exact**

Now: By The method of M

$$1-\emptyset(x,y)=\int M(x,y)\,dx$$

$$\emptyset(x,y)=\int(3y+e^{x})\,dx$$

$$\emptyset(x,y)=3yx+e^{x}+g(y) \qquad *$$

$$2-N(x,y)=\frac{d}{dy}M^{*}+g'(y)$$

$$3x+cosy=3x+g'(y)$$

$$cosy=g'(y)$$

$$3-\int \quad cosydy=\int \quad g'(y)dy$$

$$siny+c=g(y)$$

$$4-\emptyset(x,y)=3yx+e^{x}+g(y) \quad *$$

$$\emptyset(x,y)=3yx+e^{x}+siny+c$$

Example:

Find the general solution for differential equation.

$$(\boldsymbol{ycosx+siny})\boldsymbol{dx}+(\boldsymbol{sinx+xcosy-siny})\boldsymbol{dy}=\mathbf{0}$$

Solution:

$$(ycosx+siny)dx+(sinx+xcosy-siny)dy=0$$

M(x,y)dx+N(x,y)dy=0

$M(x,y) = ycosx + siny \rightarrow \; M_y = cosx + cosy$

$N(x,y) = sinx + xcosy - siny \rightarrow N_x = cosx + cosy$

$$M_y = N_x$$

Now: By The method of M

$$1 - \emptyset(x,y) = \int M(x,y)dx$$

$$\emptyset(x,y) = \int (ycosx + siny)dx$$

$$\emptyset(x,y) = ysinx + xsiny + g(y) \qquad *$$

$$2 - N(x,y) = \frac{d}{dy} M^* + g'(y)$$

$$sinx + xcosy - siny = sinx + xcosy + g'(y)$$

$$-siny = g'(y)$$

$$3 - \int -sinydy = \int \quad g'(y)dy$$

$$cosy + c = g(y)$$

$$\emptyset(x,y) = ysinx + xsiny + g(y) \quad *$$

$$\emptyset(x,y) = ysinx + xsiny + cosy + c$$

Exercise:

Find the general solution for differential equation.

$1 - ye^{xy}dx + xe^{xy}dy = 0$

$2 - Lny\, dx + \frac{x}{y}dy = 0$

$3 - e^x cosydx + (1 - e^x)sinydy = 0$

$4 - (y\cos x + 2xe^{y})dx + (\sin y + x^{2}e^{y} + 2)dy = 0$

4-Not Exact differential equation:

M(x,y)dx + N(x,y) dy= 0

$M_y = \emptyset_x$,

$N_x = \emptyset_y$

$M_y \neq N_x$

Method of solution

I(x, y) $(M(x,y)dx + N(x,y)dy)$=0 is Exact

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The method of finding an integration factor

Case one:

$if \quad \frac{M_y - N_x}{N} = g(x) \qquad \rightarrow I(x,y) = e^{\int g(x)dx}$

Example:

$$(x^2 + y^2 + x)dx + xydy = 0$$

Solution:

M(x,y) dx + N(x,y) dy=0

$M_y = 2y$

$N_x = y$

$M_y \neq N_x$

Now:

$if \quad \frac{M_y - N_x}{N} = g(x) \qquad \rightarrow I(x,y) = e^{\int g(x)dx}$

$\frac{M_y - N_x}{N} = \frac{2y - y}{xy} = \frac{y}{xy} = \frac{1}{x} = g(x)$

$I(x,y) = e^{\int g(x)dx} = e^{\int \frac{1}{x}dx} = e^{Lnx} = x$

$I(x,y) = x.[(x^2 + y^2 + x)dx + xydy = 0]$

$(x^3 + xy^2 + x^2)dx + yx^2dy = 0 \quad is$ Exact

$M_y = 2xy$

$N_x = 2xy$

$M_y = N_x$

Then is solution by method of **Exact differential equation**

$$(x^3 + xy^2 + x^2)dx + yx^2dy = 0$$

$$M_y = N_x$$

Now: By The method of M

$$1 - \emptyset(x,y) = \int M(x,y)dx$$

$$\emptyset(x,y) = \int (x^3 + xy^2 + x^2)dx$$

$$\emptyset(x,y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + g(y) \qquad *$$

$$2 - N(x,y) = \frac{d}{dy}M^* + g'(y)$$

$$yx^2 = \frac{2x^2y}{2} + g'(y)$$

$$yx^2 = x^2y + g'(y)$$

$$0 = g'(y)$$

$$3 - \int 0\,dy = \int \quad g'(y)dy$$

$$c = g(y)$$

$$\emptyset(x,y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + g(y) \qquad *$$

$$\emptyset(x,y) = \frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} + c \qquad *$$

Case two:

$if \quad \frac{M_y - N_x}{M} = h(y) \rightarrow I(x,y) = e^{-\int h(y)dy}$

$$y^2dx + xydy = 0$$

<u>Example:</u>

$$y^2dx + xydy = 0$$

Solution:

M(x,y)dx+N(x,y)dy=0

$M_y = 2y$

$N_x = y$

$M_y \neq N_x$

Now

$if \quad \frac{M_y - N_x}{M} = h(y) \rightarrow I(x,y) = e^{-\int h(y)dy}$

$$\frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y} = h(y)$$

$I(x,y) = e^{-\int h(y)dy} = e^{-\int \frac{1}{y}dy} = e^{-Lny} = e^{Lny^{-1}} = \frac{1}{y}$

$I(x,y) = \frac{1}{y}. \quad (y^2dx + xydy = 0)$

$$ydx + xdy = 0$$

is **Exact**

$M_y = 1$

$N_x = 1$

$M_y = N_x$

Then is solution by method of **Exact differential equation**

Case three:

$M = yf(x,y) \quad and \;\; N = xg(x,y) \rightarrow I(x,y) = \frac{1}{xM - yN}$

<u>Example:</u>

$$\boldsymbol{y}' = \frac{\boldsymbol{xy^2 - y}}{\boldsymbol{x}}$$

Solution:

$$\frac{dy}{dx} = \frac{xy^2 - y}{x}$$

$(\boldsymbol{xy^2} - \boldsymbol{y})dx - xdy = 0$

$M = yf(x, y) \quad and \;\; N = xg(x, y)$

M(x,y)dx+N(x,y)dy=0

$M_y = 2xy - 1$

$N_x = -1$

$M_y \neq N_x$

Now:

$$M = yf(x, y) \quad and \;\; N = xg(x, y) \rightarrow I(x, y) = \frac{1}{xM - yN}$$

$I(x, y) = \dfrac{1}{xM - yN} = \dfrac{1}{x(\boldsymbol{xy^2} - \boldsymbol{y}) - y(-x)} = \dfrac{1}{x^2y^2}$

Now:

$\dfrac{1}{x^2y^2}\Big((\boldsymbol{xy^2} - \boldsymbol{y})dx - xdy\Big) = 0 \;\; is$ Exact

$$\left(\frac{\boldsymbol{1}}{\boldsymbol{x}} - \frac{1}{x^2y}\right)dx - \frac{1}{xy^2}dy = 0$$

Then is solution by method of **Exact differential equation**

Exercise:

Find the general solution for differential equation.

$\mathbf{1} - \boldsymbol{y}(y + 2x - 2)dx - 2(x + y)dy = 0$

$\mathbf{2} - (y^2 - 3y - x)dx + (2y - 3)dy = 0$

$3 - (2y + 3xy^2)dx + (x + 2x^2y)dy = 0$

$4 - (x^2)dx + 2ydy = 0$

6-linear differential equation:

$$y' + p(x)y = q(x)$$

The method

$$y' + p(x)y = q(x)$$

$$I[y' + p(x)y] = \frac{d}{dx}y.I$$

$$Iy' + Ip(x)y = Iy' + y\frac{dI}{dx} \quad \div y', y$$

$$Ip(x) = \frac{dI}{dx}$$

$$\int \frac{dI}{I} = \int p(x)dx$$

$$ln|I| = \int p(x)dx$$

$$I = e^{\int p(x)dx}$$

<u>Example:</u>

Find the general solution for differential equation.

$y' + 3y = e^{-2x}$

Solution:

$$y' + 3y = e^{-2x}$$

$$y' + p(x)y = q(x) \ \ is\ linear$$

$$I = e^{\int p(x)dx}$$

$$I = e^{\int 3dx} = e^{3x}$$

$$e^{3x}\left(y' + 3y = e^{-2x}\right)$$

$$y'e^{3x} + 3e^{3x}y = e^{x}$$

$$\frac{d}{dx}(y\,e^{3x}) = e^{x}$$

$$\int \quad \frac{d}{dx}(ye^{3x})dx = \int \quad e^{x}dx$$

$$ye^{3x} = e^{x} + A$$

$$y = \frac{e^{x} + A}{e^{3x}}$$

<u>Example:</u>

Find the general solution for differential equation.

$y' - 5y = x^2$

Solution:

$$y' - 5y = x^2$$

$$y' + p(x)y = q(x)$$

$$I = e^{\int p(x)dx}$$

$$I = e^{\int -5dx} = e^{-5x}$$

$$e^{-5x}\left(y' - 5y = x^2\right)$$

$$y'e^{-5x} - 5e^{-5x}y = x^2e^{-5x}$$

$$\frac{d}{dx}(ye^{-5x}) = x^2e^{-5x}$$

$$\int \frac{d}{dx}(ye^{-5x})dx = \int x^2\, e^{-5x}dx$$

$$ye^{-5x} = \frac{x^2}{5}e^{-5x} + \frac{2x}{25}e^{-5x} + \frac{2}{125}e^{-5x} + A$$

Derivative	+-	Integral
x^2	+	e^{-5x}
2x	-	$\frac{-1}{5}e^{-5x}$
2	+	$\frac{1}{25}e^{-5x}$
0	-	$\frac{-1}{125}e^{-5x}$

Example:

Find the general solution for differential equation.

$y'x^2 + 2xy = 1$

Solution:

$$y'x^2 + 2xy = 1 \quad \div x^2$$

$$y' + \frac{2y}{x} = \frac{1}{x^2}$$

$$y' + p(x)y = q(x)$$

$$I = e^{\int p(x)dx}$$

$$I = e^{2\int \frac{1}{x}dx} = e^{2lnx} = e^{lnx^2} = x^2$$

$$x^2y' + 2xy = 1$$

$$\frac{d}{dx}(yx^2) = 1$$

$$\int \quad \frac{d}{dx}(yx^2)dx = \int 1\,dx$$

$$yx^2 = x + A$$

$$y = \frac{x + A}{x^2}$$

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<u>Exercise:</u>

Find the general solution for differential equation.

$1 - y' = cscx - ycotx$

$2 - y'x^3 + 2x^2y = 1$

$3 - y' + ycotx = 5e^{cosx}$

$4 - y' + y = sinx$

<u>7-Bernoulli equation:</u>

$$y' + p(x)y = q(x)\, y^n \qquad n \neq 0,1$$

Example:

Find the general solution for differential equation.

$y' - y = x\, y^2$

Solution:

$y' - y = x\, y^2 \quad * \ y^{-2}$

$y^{-2}y' - y^{-1} = x \qquad *$

$Let\ w = (y)^{-1} \quad \rightarrow y = \dfrac{1}{w}$

$w' = -y^{-2}y'$

$-w' = y^{-2}y'$

By

$y^{-2}y' - y^{-1} = x \qquad *$

$-w' - w = x$

$w' + w = -x \qquad Linear$

$$I = e^{\int p(x)dx}$$

$$I = e^{\int dx} = e^x$$

$$e^x(w' + w = -x)$$

$$w'e^x + we^x = -xe^x$$

$$\frac{d}{dx}(we^x) = -xe^x$$

$$\int \quad \frac{d}{dx}(we^x)dx = -\int \quad xe^x dx$$

$$we^x = e^x - xe^x + c$$

$$w = 1 - x + \frac{c}{e^x}$$

$$y = \frac{1}{1 - x + \frac{c}{e^x}}$$

Derivative	+-	Integral
x	+	e^x
1	-	e^x
0	+	e^x

Example:

Find the general solution for differential equation.

$$\boldsymbol{y' - \frac{1}{x}y = x^3\ \ y^3}$$

Solution:

$$y' - \frac{y}{x} = x^3\ \ y^3 \quad * \quad y^{-3}$$

$$y^{-3}y' - \frac{y^{-2}}{x} = \ x^3 \qquad *$$

$$Let\ w = y^{-2} \quad \to y = \sqrt{\frac{1}{w}}$$

$$w' = -2y^{-3}y' \qquad \div (-2)$$

$$\frac{w'}{-2} = y^{-3}y'$$

By

$$y^{-3}y' - \frac{y^{-2}}{x} = \ x^3 \qquad *$$

$$\frac{w'}{-2} - \frac{w}{x} = x^3 \qquad * -2$$

$$w' + \frac{2w}{x} = -2x^3 \qquad Linear$$

$$I = e^{\int p(x)dx}$$

$$I = e^{\int \frac{2}{x}dx} = e^{2lnx} = e^{lnx^2} = x^2$$

$$x^2\left(w' + \frac{2w}{x} = -2x^3\right)$$

$$x^2w' + 2\text{x}w = -2x^5$$

$$\frac{d}{dx}(x^2w) = -2x^5$$

$$\int \quad \frac{d}{dx}(x^2w)dx = \int \quad -2x^5dx$$

$$x^2w = \frac{-x^6}{3} + c$$

$$w = \frac{-x^4}{3} + \frac{c}{x^2}$$

$$y = \sqrt{\frac{1}{\frac{-x^4}{3} + \frac{c}{x^2}}} = \sqrt{\frac{-3}{x^4} + \frac{x^2}{c}}$$

<u>Example:</u>

Find the general solution for differential equation.

$$\boldsymbol{2xyy' = y^2 - 2x^3}$$

Solution.

$$\boldsymbol{2xyy' = y^2 - 2x^3}$$

$$\mathbf{2xyy' - y^2 = -2x^3 \quad \div 2xy}$$

$$\mathbf{y' - \frac{y}{2x} = \frac{-x^2}{y} \quad * y}$$

$$\mathbf{yy' - \frac{y^2}{2x} = -x^2}$$

$$w = y^2 \quad \rightarrow y = \sqrt{w}$$

$$w' = 2yy' \quad \rightarrow \frac{w'}{2} = yy'$$

$$\mathbf{yy' - \frac{y^2}{2x} = -x^2}$$

$$\frac{w'}{2} - \mathbf{\frac{w}{2x} = -x^2 \quad * (2)}$$

$$w' - \mathbf{\frac{w}{x} = -2x^2}$$

$$I = e^{\int p(x)dx}$$

$$I = e^{-\int \frac{1}{x}dx} = e^{-lnx} = e^{lnx^{-1}} = \frac{1}{x}$$

$$\frac{1}{x}\left(w' - \mathbf{\frac{w}{x} = -2x^2}\right)$$

$$\frac{w'}{x} + \mathbf{\frac{w}{x^2} = -2x}$$

$$\frac{d}{dx}\left(\frac{w}{x}\right) = -2x$$

$$\int \frac{d}{dx}\left(\frac{w}{x}\right) dx = \int -2x\, dx$$

$$\frac{w}{x} = -x^2 + c$$

$$y = \sqrt{\frac{-x^2 + c}{x}}$$

<u>Exercise:</u>

Find the general solution for differential equation.

$$\mathbf{1 - y' + xy = \frac{x}{y}}$$

$$\mathbf{2 - y' = y - xy^3 e^{-2x}}$$

$$\mathbf{3 - y' sinx - ycosx + y^2 = 0}$$

<u>Orthogonal trajectories:</u>

In mathematics an orthogonal trajectory is a curve which intersects any curve of a given pencil of planar curves orthogonally

<u>Example:</u>

If the slope of curve is 6xy find the equation of the curve if the curve throw the point (2,1).

Solution.

$$\frac{dy}{dx} = 6xy$$

$$\int \frac{dy}{y} = \int 6xdx$$

$$Lny = \frac{6x^2}{2} + c$$

$$Lny = 3x^2 + c$$

$$Ln1 = 3(2)^2 + c \qquad Ln1 = 0$$

$c = -12$

$Lny = 3x^2 - 12$

Example:

Find the orthogonal trajectories of the family of the curve $\boldsymbol{y = cx^2}$.

Solution:

$y = cx^2 \quad *$

$$c = \frac{y}{x^2}$$

$y' = 2cx$

$$y' = 2\frac{y}{x^2}x$$

$$y' = 2\frac{y}{x}$$

$$y' orthogonal = \frac{-x}{2y}$$

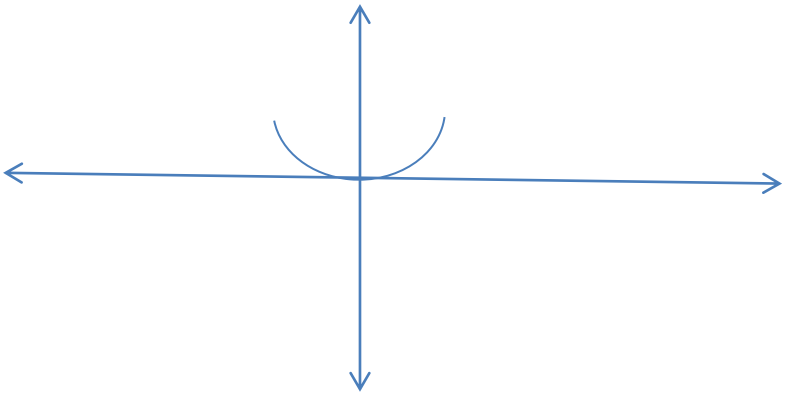
$$\frac{dy}{dx} = \frac{-x}{2y}$$

$$\int \quad 2ydy = -\int xdx$$

$$y^2 = -\frac{x^2}{2} + A$$

$$y^2 + \frac{x^2}{2} = A$$

$$\frac{y^2}{A} + \frac{x^2}{2A} = 1 \quad Ellipse$$



Example:

Find the orthogonal trajectories of the family of the circle and the center is point of origin.

Solution:

By the circle point of origin the equation is

$$x^2 + y^2 = c$$

$$2x + 2yy' = 0$$

$$x + yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y' orthogonal = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

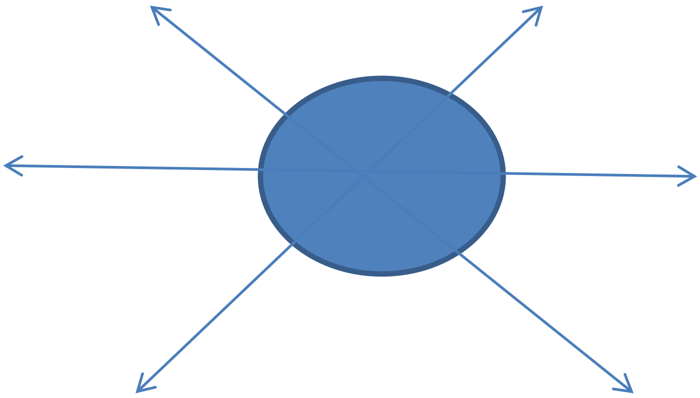
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$Ln|y| = Ln|x| + A$$

$$y = xe^{A}$$

$$y = kx$$

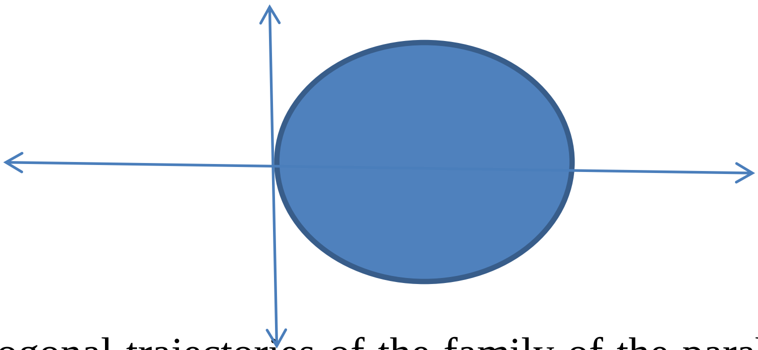
Is the equation of line



Exercise:

1- Find the orthogonal trajectories of the family of the circle throw the point of origin and the center in X-axsis.

$(x-c)^2 + y^2 = c^2$



2- Find the orthogonal trajectories of the family of the parabola throw the point of origin in X-axis.

$y^2 = 4cx$

3- Find the orthogonal trajectories of the family of the curve $y^2 = cx^3$

4- Find the orthogonal trajectories of the family of the curve $x - 4y = c$

5- Find the orthogonal trajectories of the family of the circle throw the point of origin and the center in y-axsis.

$x^2 + (y-c)^2 = c^2$

Application physics

Example:

Electrical Circuit consisting of resistance R and generator (coil) self generated factor (L) have been connected with battery her voltage (E) find the current (I) for this circuit if i=0 and t=0.

Solution.

$$L\frac{di}{dt}+Ri=E$$

$$\frac{di}{dt}+\frac{R}{L}i=\frac{E}{L} \qquad *$$

$$p(x)=\frac{R}{L}\,, \qquad q(x)=\frac{E}{L}$$

$$I.F=e^{\int\frac{R}{L}dt}=e^{\frac{R}{L}t}$$

$$\frac{di}{dt}e^{\frac{R}{L}t}+\frac{R}{L}ie^{\frac{R}{L}t}=\frac{E}{L}e^{\frac{R}{L}t}$$

$$\int\frac{di}{dt}\left(ie^{\frac{R}{L}t}\right) \quad =\int\frac{E}{L}\;e^{\frac{R}{L}t}dt$$

$$ie^{\frac{R}{L}t}=\frac{E}{L}\int \qquad e^{\frac{R}{L}t}dt$$

$$ie^{\frac{R}{L}t}=\frac{E}{L}\;e^{\frac{R}{L}t}+c$$

$$i=\frac{\frac{E}{L}e^{\frac{R}{L}t}+c}{e^{\frac{R}{L}t}} \qquad i=0,t=0, \qquad c=\frac{-E}{R}$$

$$i=\frac{E}{R}\left(1-e^{-\frac{R}{L}t}\right)$$

Reduction the order of the second order differential equation to first order ordinary differential equation

If there exist second order ordinary differential equation there are two cases.

Case one y is deleted:

Let $y' = p \; , y'' = \frac{dp}{dx} = p'$

Example:

Reduction the order of the second order ordinary differential equation.

$\boldsymbol{y'' + y' + x = 0}$

Solution.

$y'' + y' + x = 0$

Since y is deleted

Let $y' = p \; , y'' = \frac{dp}{dx} = p'$

$p' + p = -x$

$p' + p = -x$ its linear

$I.F = e^{\int dx} = e^x$

$e^x(p' + p = -x)$

$e^x p' + e^x p = -xe^x$

$$\int (pe^x)' = -\int xe^x dx$$

Derivative	+-	Integral
x	+	e^x
1	-	e^x
0	+	e^x

$pe^x = xe^x - e^x + c$

$$p = \frac{xe^x - e^x + c}{e^x} = x - 1 + c\frac{1}{e^x}$$

$$y' = x - 1 + c\frac{1}{e^x}$$

$$\int \quad y' = \int \quad \left(x - 1 + c\frac{1}{e^x}\right) dx$$

$$y = \frac{x^2}{2} - x - ce^{-x} + A$$

<u>Case two x is deleted:</u>

Let $y' = p \ , y'' = p\frac{dp}{dy}$

<u>Example:</u>

Reduction the order of the second order ordinary differential equation.

$\boldsymbol{yy'' + y^2 = 2(y')^2}$

Solution.

$yy'' + y^2 = 2(y')^2$

Since x is deleted

Let $y' = p \ , y'' = p\frac{dp}{dy}$

$$yp\frac{dp}{dy} + y^2 = 2(p)^2$$

$$yp\frac{dp}{dy} = 2(p)^2 - y^2$$

$$\frac{dp}{dy} = \frac{2p}{y} - \frac{y}{p} \quad *$$

Suppose $z = \frac{p}{y} \rightarrow p = yz$

$$\frac{dp}{dy} = y\frac{dz}{dy} + z \quad substitute\ in *$$

$$y\frac{dz}{dy} + z = 2z - \frac{1}{z}$$

$$y\frac{dz}{dy} = \frac{z^2 - 1}{z} \quad V.S$$

$$\int \frac{dy}{y} = \int \frac{z}{z^2 - 1}\ dz$$

$$lny = \frac{1}{2}Ln|z^2 - 1| + c$$

$$lny = \frac{1}{2}Ln|(z)^2 - 1| + c$$

$$lny = \frac{1}{2} Ln|(z)^2 - 1| + c$$

$$lny = Ln((z)^2 - 1)^{\frac{1}{2}} + c$$

$$y = ((z)^2 - 1)^{\frac{1}{2}} + e^c$$

$$y^2 = \left(((z)^2 - 1)^{\frac{1}{2}} + A\right)^2$$

$$y^2 = ((z)^2 - 1)^{\frac{2}{2}} + 2(A(z)^2 - 1)^{\frac{1}{2}} + A^2$$

$$y^2 = z^2 + 2(A(z)^2 - 1)^{\frac{1}{2}} + A^2 - 1$$

Example:

Reduction the order of the second order ordinary differential equation.

$$(\boldsymbol{x} - \mathbf{1})\boldsymbol{y}'' + \boldsymbol{y}' - (\boldsymbol{x} - \mathbf{1})^{\mathbf{2}} = \mathbf{0}$$

Solution.

$$(x - 1)y'' + y' - (x - 1)^2 = 0$$

Since y is deleted

Let $y' = p$, $y'' = \frac{dp}{dx} = p'$

$$(x - 1)p' + p - (x - 1)^2 = 0$$

$$(x - 1)p' + p = (x - 1)^2$$

$$p' + \frac{1}{x - 1} p = (x - 1)$$

$$I.F = e^{\int \frac{1}{x-1} dx} = e^{Ln|x-1|} = x - 1$$

$$(x-1)\left(p' + \frac{1}{x-1}p = (x-1)\right)$$

$$(x-1)p' + p = (x-1)^2$$

$$\int ((x-1)p)' = \int (x-1)^2\, dx$$

$$(x-1)p = \frac{(x-1)^3}{3} + c$$

$$p = \frac{(x-1)^2}{3} + \frac{c}{x-1}$$

$$y' = \frac{(x-1)^2}{3} + \frac{c}{x-1} \qquad Exc$$

<u>Exercise:</u>

Reduction the order of the second order differential equation.

$\mathbf{1} - y'' + yy' = 0$

$\mathbf{2} - xy'' + y' = x^2$

$\mathbf{3} - y'' = \frac{4}{3}yy' \qquad y(2) = 1\ , \qquad y'(2) = \frac{2}{3}$

$\mathbf{4} - y'' = \frac{y'}{x} \qquad y(1) = 3\ , \qquad y'(1) = 1$

n – order differential equation:

$1 - y'' + 3y' + y = \text{sinx}.$

linear D. E non homogeneous second order

$2 - y''' + y'' + y' + y = 0$

linear D. E homogeneous third order

$3 - y^5 + 3y^4 + y'' + y = e^x.$

linear D. E non homogeneous fifth order

Example:

$\boldsymbol{y = c_1 sinx + c_2 cosx}$ **is a solution of** $\mathbf{y'' + y = 0}$

Solution.

$y_1 = sinx$

$y'_1 = cosx$

$y''_1 = -sinx$

By $\mathbf{y'' + y = 0}$

$-sinx + sinx = 0$

$y_2 = cosx$

$y'_2 = -sinx$

$y''_2 = -cosx$

By $\mathbf{y'' + y = 0}$

$-cosx + cosx = 0$

$y = c_1 sinx + c_2 cosx$

$y' = c_1 cosx - c_2 sinx$

$y'' = -c_1 sinx - c_2 cosx$

By $y'' + y = 0$

$-c_1 sinx - c_2 cosx + c_1 sinx + c_2 cosx = 0$

Yes: $y = c_1 sinx + c_2 cosx$ is a solution of $y'' + y = 0$

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Exercise:

1) $\boldsymbol{y = c_1 e^x + c_2 e^x}$ **is a solution of**

$\mathbf{y'' - y = 0}$

2) $\boldsymbol{y = c_1 e^{3x} + c_2 e^{-2x}}$ **is a solution of**

$\mathbf{y'' - y' + 6y = 0}$

Wronskian determinant

Let $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ be a function that differentiable in the interval $I = [a, b]$ then the Wronskian determinant for this function.

$$w(x) = \begin{vmatrix} y_1 & y_2 \cdots & y_n \\ y'_1 & y'_2 \cdots & y'_n \\ y''_1 & y''_2 \cdots & y''_n \end{vmatrix}$$

Remark:

If the number of functions equal n then we derive the function to n-1.

Example:

Find the Wronskian determinant for set $\{x, x^5\}$

Solution:

$$w(x) = \begin{vmatrix} x & x^5 \\ 1 & 5x^4 \end{vmatrix}$$

$$= 5x^5 - x^5 = 4x^5$$

Example:

Find the Wronskian determinant for set $\{1, x, x^3\}$

Solution:

$$w(x) = \begin{vmatrix} 1 & x & x^3 \\ 0 & 1 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} = 6x$$

Example:

Find the Wronskian determinant for set $\{e^x, x^2, x\}$

Solution:

$$w(x) = \begin{vmatrix} e^x & x^2 & x \\ e^x & 2x & 1 \\ e^x & 2 & 0 \end{vmatrix}$$

$$w(x) = x\begin{vmatrix} e^x & 2x \\ e^x & 2 \end{vmatrix} - 1\begin{vmatrix} e^x & x^2 \\ e^x & 2 \end{vmatrix}$$

$$w(x) = x(2e^x - 2xe^x) - (2e^x - x^2e^x)$$

$$w(x) = 2xe^x - 2x^2e^x - 2e^x + x^2e^x$$

$$w(x) = e^x(2x - 2x^2 - 2 + x^2)$$

$w(x) = e^x(-x^2 + 2x - 2)$

Example:

Find the Wronskian determinant for set $\{x^2, x^3, e^{-x}, e^x\}$

Solution:

$$w(x) = \begin{vmatrix} e^x & e^{-x} & x^2 & x^3 \\ e^x & -e^{-x} & 2x & 3x^2 \\ e^x & e^{-x} & 2 & 6x \\ e^x & -e^{-x} & 0 & 6 \end{vmatrix}$$

Step-1-

$$w(x) = \begin{vmatrix} e^x & e^{-x} & x^2 & x^3 \\ 0 & -2e^{-x} & 2x - x^2 & 3x^2 - x^3 \\ e^x & e^{-x} & 2 & 6x \\ e^x & -e^{-x} & 0 & 6 \end{vmatrix} r_2 - r_1 \to r_2$$

Step-2-

$$w(x) = \begin{vmatrix} e^x & e^{-x} & x^2 & x^3 \\ 0 & -2e^{-x} & 2x - x^2 & 3x^2 - x^3 \\ 0 & 0 & 2 - x^2 & 6x - 3 \\ e^x & -e^{-x} & 0 & 6 \end{vmatrix} r_3 - r_1 \to r_3$$

Step-3-

$$w(x) = \begin{vmatrix} e^x & e^{-x} & x^2 & x^3 \\ 0 & -2e^{-x} & 2x - x^2 & 3x^2 - x^3 \\ 0 & 0 & 2 - x^2 & 6x - 3 \\ 0 & -2e^{-x} & 0 - x^2 & 6 - x^3 \end{vmatrix} r_4 - r_1 \to r_4$$

Step-4-

$$w(x) = e^x \begin{vmatrix} -2e^{-x} & 2x - x^2 & 3x^2 - x^3 \\ 0 & 2 - x^2 & 6x - 3 \\ -2e^{-x} & -x^2 & 6 - x^3 \end{vmatrix}$$

<u>Step-5-</u>

$$w(x) = e^{x}\begin{vmatrix} -2e^{-x} & 2x - x^{2} & 3x^{2} - x^{3} \\ 0 & 2 - x^{2} & 6x - 3 \\ 0 & -2x & 6 - 3x^{2} \end{vmatrix} r_3 - r_1 \to r_3$$

Step-6-

$$w(x) = e^{x}\begin{vmatrix} -2e^{-x} & 2x - x^{2} & 3x^{2} - x^{3} \\ 0 & 2 - x^{2} & 6x - 3 \\ 0 & -x^{2} & 6 - x^{3} \end{vmatrix} r_3 + r_1 \to r_3$$

$$w(x) = e^{x}[-2e^{-x}]\begin{vmatrix} 2 - x^{2} & 6x - 3 \\ -x^{2} & 6 - x^{3} \end{vmatrix}$$

$$w(x) = e^{x}[-2e^{-x}[(2 - x^{2})(6 - x^{3}) - (-x^{2})(6x - 3)]]$$

$$w(x) = e^{x}\left[-2e^{-x}[(12 - 2x^{3} - 6x^{2} + x^{5}) - (-6x^{3} + 3x^{2})]\right]$$

$$w(x) = e^{x}[-2e^{-x}[(12 - 2x^{3} - 6x^{2} + x^{5}) + 6x^{3} - 3x^{2})]]$$

$$w(x) = e^{x}[-2e^{-x}[(12 + 4x^{3} - 9x^{2} + x^{5})]]$$

$$w(x) = e^{x}[-24e^{-x} - 8x^{3}e^{-x} + 18x^{2}e^{-x} - 2x^{5}e^{-x}]$$

$$w(x) = -24 \quad -8x^{3} \quad +18x^{2} \quad -2x^{5}$$

Exercise:

1) Find the Wronskian determinant for set $\{sinx, e^x, cosx, sinhx, coshx\}$

2) Find the Wronskian determinant for set $\{sinhx, coshx, -sinhx, cosx\}$

3)Find the Wronskian determinant for set $\{x^2, x^{-2}, x^3\}$

Linearly non Homogenous Differential equation of order n.

$L(y) = \emptyset(x) \quad *$

Theorem

Let y_p particular solution of the linearly non Homogenous Differential equation of order n.

(*) and let y_h is the general solution of $L(y) = 0$

Then the general solution of (*) is $y_p + y_h$

Proof:

$L(y_p + y_h) = \emptyset(x)$

$L(y) = \emptyset(x)$

$L(y) = 0 \quad is\ linear$

$L(y_p + y_h) = L(y_p) + L(y_h)$

$L(y_p + y_h) = \emptyset(x) + 0 = \emptyset(x)$

$y_p + y_h$ is a solution for (*) $\vee \quad L(y) = \emptyset(x)$

Let y be the general solution for $L(y) = \emptyset(x)$

$z = y - y_p$

$y = z + y_p$

$L(z) = L(y) + L(y_p)$

$L(z) = \emptyset(x) - \emptyset(x)$

$L(z) = 0$

$\therefore z \; is \; solution \; for \; L(y) = 0$

$z = y_h$

An existence and uniqueness theorem

Let $f(X), \; a_0\,(x), a_1\,(x) \; .... \; a_n(x)$

be a continuous function on the interval $I = [a, b]$

suppose that $x_0 \in I \; and \; c_0 \; , \; c_1 \; .... \; , c_{n-1}$ for n arbitrary constant in I then uniquenees solution $y = y(x)$ is exist and define on I.

which is a solution of the initial value probleme

$$y^n + a_0\,(x)y^{n-1} + \cdots \quad a_{n-1}\,(x)y' + \quad a_n(x)y = 0$$

Which

$y(x_0) = c_0$

$y'(x_0) = c_1$

.

.

.

$y^{n-1}(x_0) = c_{n-1}$

Remarks:

1-The solution is exist and define for this solution

2- if exist condition exist only one solution

3- if not exist condition exist infinite solution

Example:

Find the unique solution for the initial value problem

$\boldsymbol{y'' + y = 0}.\qquad \boldsymbol{y(0) = 0}, \quad \boldsymbol{y'(0) = 1}$

Proposition:

Let $\{y_1,\ y_2, \ldots \qquad , y_n\}$ the set of solution

The Linearly non Homogenous Differential equation of order n is linearly dependent iff

$w(x) = 0, forall\ x \in I$

Proof:→

Suppose $w(x) = 0, forall\ x \in I$

$$\exists\ c_1\ , c_2\ \ldots .\ c_n\ \ not\ all = 0$$

$$\begin{vmatrix} y_1 & y_2 \cdots & y_n \\ y'_1 & y'_2 \cdots & y'_n \\ \vdots & \vdots & \vdots \\ y''_1 & y''_2 \cdots & y''_n \end{vmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

We transform these matrices to equation

$c_1\ y_1(x_0) + c_2\ y_2(x_0) + \cdots + c_n\ y_n(x_0) = 0$

$c_1\, y'_1(x_0) + c_2\, y'_2(x_0) + \cdots . + y'_n(x_0) = 0$

.

.

$c_1\, y_1{}^{n-1}(x_0) + c_2\, y_2{}^{n-1}(x_0) + +c_n\, y_n{}^{n-1}(x_0) = 0$

Suppose

$y(x) = \; c_1\, y_1 + c_2\, y_2 + \cdots + c_n\, y_n$

It is a solution for Initial value problem

$y^n + \; a_0(x) y_1{}^{n-1} \ldots + a_{n-1}(x) y^{'} + a_n(x) y = 0$

With

$y(x_0) = 0\, , y'(x_0) = 0\, , \ldots y^{n-1}(x_0) = 0$

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Remark:

If Linearly non Homogenous Differential equation of order n is linearly independent is

$w(x) \neq 0, forall\ x \in I.$

Second order linear homogeneous D.E with constant coefficients

Definition:

Algebra equation is

$\lambda^2 + a\lambda + b = 0 \qquad (1)$

We get them from differential equation

is $y'' + ay' + by = 0 \qquad (2)$

and put y'', y', y a place $\lambda'', \lambda', \lambda^0 = 1$

for the equation (1)

$\lambda'' = y'', \lambda' = y', \lambda^0 = y = 1$

This is characteristic equation

Example:

Find the characteristic equation

$\boldsymbol{y''' + 3y'' - 5y' + 6y = 0}$

Solution.

$\lambda^3 + 3\lambda^2 - 5\lambda + 6 = 0$ characteristic equation

Example:

Find the characteristic equation

$\boldsymbol{y'' + 2y' + 5y = 0}$

Solution.

$\lambda^2 + 2\lambda + 5 = 0$ characteristic equation

Example:

Find the characteristic equation

$\boldsymbol{y'' - 5y = 0}$

Solution.

$\lambda^2 - 5 = 0$ characteristic equation

Example:

Find the characteristic equation

$$\boldsymbol{y'' - x^2 y = 0}$$

Solution.

Do not have characteristic equation since have the variable x.

Method to solve Algebra equation is

$$\boldsymbol{\lambda^2 + a\lambda + b = 0}$$

The solve of characteristic equation

$$\lambda^2 + a\lambda + b = 0 \qquad (1)\ is$$

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0$$

The solution of (1)

$$Ax^2 + Bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}$$

$$\lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

There are three cases to find λ

Case one:

If case $a^2 - 4b > 0 \rightarrow \sqrt{a^2 - 4b} > 0$

$\therefore \lambda_1 \neq \lambda_2$ *Real numbers*

Then there exist two solution for independent equation $w(x) \neq 0$ for equation (2)

$y_1 = e^{\lambda_1 x}, \qquad y_2 = e^{\lambda_2 x}$

$y = c_1 y_1 + c_2 y_2$ *this is general equation*

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_1 x}$ *this is general equation*

Case two:

If case $a^2 - 4b = 0 \rightarrow \sqrt{a^2 - 4b} = 0$

$\therefore \lambda_1 = \lambda_2 = \frac{-a}{2}$ *Real numbers*

Then there exist two solution for independent equation $w(x) \neq 0$ for equation (2)

$y_1 = e^{\lambda_1 x} = e^{\frac{-a}{2}x}$

$y_2 = e^{\lambda_2 x} = x e^{\frac{-a}{2}x}$

$y = c_1 y_1 + c_2 y_2$ *this is general equation*

$y = c_1 e^{\frac{-a}{2}x} + c_2 x e^{\frac{-a}{2}x}$ *this is general equation*

Case three:

If case $a^2 - 4b < 0$

$$\sqrt{a^2 - 4b} = \sqrt{-(4b - a^2)} = \sqrt{-1}\sqrt{4b - a^2}$$

$$= i\sqrt{4b - a^2}$$

$$\therefore \quad \lambda_1 \neq \lambda_2 = \frac{-a}{2} \text{ complex number}$$

$$\lambda_1 = \frac{-a + i\sqrt{4b - a^2}}{2} = \frac{-a}{2} + \frac{i\sqrt{4b - a^2}}{2}$$

$$\lambda_2 = \frac{-a - i\sqrt{4b - a^2}}{2} = \frac{-a}{2} - \frac{i\sqrt{4b - a^2}}{2}$$

Assume

$$q = \frac{\sqrt{4b - a^2}}{2} \quad , p = \frac{-a}{2}$$

$$\therefore \quad \lambda_1 = p + iq \quad , \lambda_2 = p - iq$$

Then there exist two solution for independent equation $w(x) \neq 0$ for equation (2)

$$y_1 = e^{\lambda_1 x} = e^{(p+iq)x} = e^{px}.\ e^{iqx} = e^{px}(cosqx + isinqx)$$

$$y_2 = e^{\lambda_2 x} = e^{(p-iq)x} = e^{px}.\ e^{-iqx}$$
$$= e^{px}\big(cos(-qx) + isin(-qx)\big)$$
$$= e^{px}\big(cosqx - isin(qx)\big)$$

Then there exist two solution for independent equation $w(x) \neq 0$ for equation (2)

$$y = c_1 y_1 + c_2 y_2 \quad \text{this is general equation}$$

$$y = c_1 e^{px}(cosqx + isinqx) + c_2 e^{px}(cosqx - isinqx)$$

$$y = e^{px}[(c_1 + c_2)cosqx + (c_1 - c_2)isinqx]$$

$y = e^{px} Acosqx + Bisinqx \quad this\ is\ general\ equation$

<u>Example:</u>

Find $\boldsymbol{is\ the\ general\ equation}$

$\boldsymbol{y'' - 3y' + 2y = 0 \quad y(0) = 1 \quad y'(0) = 2}$

$\boldsymbol{Solution}.$

$y'' - 3y' + 2y = 0$

$\lambda^2 - 3\lambda + 2 = 0$ characteristic equation

$(\lambda - 2)(\lambda - 1) = 0$

$\lambda_1 = 2, \quad \lambda_2 = 1 \quad \rightarrow \quad \lambda_1 \neq \lambda_2 \ Real$

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_1 x}$

$y = c_1 e^{2x} + c_2 e^{x} \qquad y(0) = 1$

$y' = 2c_1 e^{2x} + c_2 e^{x} \qquad y'(0) = 2$

$1 = c_1 + c_2$

$\underline{\pm 2 = \pm 2c_1 \pm c_2}$

$-1 = -c_1 \rightarrow c_1 = 1, \qquad c_2 = 0$

$y = c_1 e^{2x} \quad this\ is\ general\ equation$

<u>Example:</u>

Find $\boldsymbol{is\ the\ general\ equation}$

$\boldsymbol{y'' - 3!\,y' + 8y = 0}$

$\boldsymbol{Solution}.$

$y'' - 3!\,y' + 8y = 0$

$\lambda^2 - 6\lambda + 8 = 0$ characteristic equation

$(ƛ-4)(ƛ-2)=0$

$ƛ_1=4,\quad ƛ_2=2 \quad \rightarrow \quad ƛ_1 \neq ƛ_2 \ \ Real$

$y=c_1e^{ƛ_1x}+c_2e^{ƛ_1x}$

$y=c_1e^{4x}+c_2e^{2x}$

<u>Example:</u>

Find is the general equation

$\boldsymbol{y^{''}+4y^{'}+4y=0 \quad y(0)=1 \quad y'(0)=0}$

Solution.

$y^{''}+4y^{'}+4y=0$

$ƛ^2+4ƛ+4=0$ characteristic equation

$(ƛ+2)(ƛ+2)=0$

$ƛ_1=-2=ƛ_2 \quad \rightarrow \quad Real$

$y=c_1e^{ƛ_1x}+c_2e^{ƛ_1x}$

$y=c_1e^{-2x}+xc_2e^{-2x} \qquad y(0)=1$

$y'=-2c_1e^{-2x}-2xc_2e^{x}+c_2e^{-2x} \qquad y'(0)=0$

$1=c_1$

$2=c_2$

$y=e^{-2x}+2xe^{-2x} \quad this\ is\ general\ equation$

<u>Example:</u>

Find ***is the general equation***

$\boldsymbol{y^{''}-2y^{'}+5y=0 \quad y(0)=1 \quad y'(0)=-1}$

Solution.

$y^{''} - 2y^{'} + 5y = 0$

$ƛ^2 - 2ƛ + 5 = 0$ characteristic equation

$ƛ \; = \frac{2 \pm \sqrt{4-(4\times 5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

$y_1 \; = e^{(\mathbf{1+i2})x} = e^x.\ e^{\mathbf{i2x}} = e^x(cos2x + isin2x)$

$y_1 \; = e^{(\mathbf{1-i2})x} = e^x.\ e^{\mathbf{-i2x}} = e^x(cos2x - isin2x)$

$y = e^{\mathbf{px}}(Acosqx + Bisinqx)$

$y = e^x(Acos2x + Bisin2x) \quad y(0) = 1$

$y' = e^x(-2Acos2x + 2Bisin2x) + e^x(Acos2x + Bisin2x)$

$y'(0) = -1$

$A = 1, -1 = 2B + 1 \rightarrow B = -1$

$y = e^{\mathbf{px}}(Acosqx + Bisinqx)$

$y = e^x(cos2x - isin2x)$

<u>Example:</u>

Find *is the general equation*

$\boldsymbol{y^{'''} = 0}$

Solution.

$y^{'''} = 0$

$ƛ^3 = 0$

$(ƛ - ƛ_1)(ƛ - ƛ_2)(ƛ - ƛ_3) = 0$

$ƛ_1 = ƛ_2 = ƛ_3 = 0$

$y = c_1 e^{0x} + x c_2 e^{0x} + x^2 c_3 e^{0x}$

$y = c_1 + x c_2 \quad + x^2 c_3$

Exercise:

Find the general solution for differential equation.
$1 - y''' - 2y'' - y' + 2y = 0$

$2 - y''' - y'' - y' + y = 0, y(0) = 0, y'(0) = 1, y''(0) = -1$

$3 - y''' - 3y'' + 3y' - y = 0, y(0) = 1, y'(0) = 0, y''(0) = -1$

$4 - y''' - y'' + y' - y = 0$

$5 - y'''' = 0$

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Laplace transforms:

Let $f(x)$the function defined by $[0, \infty)$**,**

The Laplace Transforms for $f(x)$ is

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x) dx, \quad s > 0$$

The domain F(s) is every value of S.

$$\int_0^\infty e^{-sx} f(x) dx = \lim_{n \to \infty} \int_0^n e^{-sx} f(x) dx$$

Properties:

$1 - L\{f(x) \pm g(x)\} = L\{f(x)\} \pm L\{g(x)\}$

$2 - L\{af(x)\} = aL\{f(x)\}$

$3 - L\{e^{ax}f(x)\} = F(s-a)$

Example:

Find Laplace transforms for $f(x) = 1$.

Or. Find Laplace transforms for $L\{1\}$

Solution.

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx$$

$$L\{1\} = \int_0^\infty e^{-sx} dx = -\frac{1}{s}e^{-sx} \quad I_0{}^\infty$$

$$L\{1\} = -\frac{1}{s}e^{-s\infty} + \frac{1}{s}e^{-s0}$$

$$L\{1\} = 0 + \frac{1}{s} = \frac{1}{s}$$

Remark:

$1-)e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

$2-)e^{\infty} = 0$

$3-)e^{0} = 1$

Example:

Find Laplace transforms for $f(x) = a$.

Find Laplace transforms for $L\{a\}$ when a is constant.

Solution.

$$L\{a\} = L\{a\ 1\} = aL\{1\} = a\frac{1}{s} = \frac{a}{s}$$

<u>Example:</u>

Find Laplace transforms for $L\{5\}$

Solution.

$$L\{5\} = \frac{5}{s}$$

<u>Example:</u>

Find Laplace transforms for $L\{-10\}$

Solution.

$$L\{-10\} = \frac{-10}{s}$$

<u>Example:</u>

Find Laplace transforms for $\boldsymbol{L\{x\}}$.

Find Laplace transforms for $\boldsymbol{f(x) = x}$.

Solution.

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx$$

$$L\{x\} = \int_0^\infty e^{-sx} x dx$$

$$L\{x\} = \left(-\frac{1}{s}xe^{-sx} - \frac{1}{s^2}e^{-sx}\right) \qquad I_0{}^\infty$$

$$L\{x\} = \left(-\frac{1}{s}\infty e^{-s\infty} - \frac{1}{s^2}e^{-s\infty}\right) - \left(-\frac{1}{s}0e^{-s0} - \frac{1}{s^2}e^{-s0}\right)$$ نبأ

$$\forall s > 0\,, e^{-sx} = 0 \ \ if\ x \to \infty$$

$L\{x\} = -\left(-\frac{1}{s^2}\right) = \frac{1}{s^2}$

x	e^{-sx}
1	$-\frac{1}{s}e^{-sx}$
0	$\frac{1}{s^2}e^{-sx}$

Example:

Find Laplace transforms for $L\{x^2\}$.

Solution.

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx$$

$$L\{x^2\} = \int_0^\infty e^{-sx} x^2 dx$$

$$L\{x^2\} = -\frac{x^2}{s}e^{-sx} - \frac{2x}{s^2}e^{-sx} - \frac{2}{s^3}e^{-sx} \qquad I_0{}^\infty$$

$\forall s > 0 , e^{-sx} = 0 \ \ if \ x \to \infty$

$$L\{x^2\} = -\left(-\frac{2}{s^3}\right) = \frac{2}{s^3}$$

Remark:

$n! = \boldsymbol{n} \times (\boldsymbol{n} - \boldsymbol{1}) \times (\boldsymbol{n} - \boldsymbol{2}) \times ....$

$3! = 3 \times \boldsymbol{2} \times \boldsymbol{1} = \boldsymbol{6}$

Exercise:

Find Laplace transforms for $L\{x^3\} = \frac{6}{s^4}$.

Exercise:

Find Laplace transforms for $L\{x^4\} = \frac{24}{s^5}$.

Remark:

The Laplace transforms for $L\{x^n\} = \frac{n!}{s^{n+1}}$.

Example:

Find Laplace transforms for $L\{e^{ax}\}$.

Solution.

$$L\{f(x)\} = \int_0^\infty e^{-sx} f(x)dx$$

$$L\{e^{ax}\} = \int_0^\infty e^{-sx} e^{ax} dx$$

$$L\{e^{ax}\} = \int_0^\infty e^{(-s+a)x} dx = \int_0^\infty e^{-(s-a)x} dx$$

$$L\{e^{ax}\} = -\frac{1}{s-a}\ e^{-(s-a)x} \quad I_0^{\ \infty}$$

$$L\{e^{ax}\} = -\left(-\frac{1}{s-a}\right) = \frac{1}{s-a}$$

Remark:

$$1 - L\{sinax\} = \frac{a}{s^2 + a^2}$$

$$2 - L\{cosax\} = \frac{s}{s^2 + a^2}$$

$$3 - L\{sinhax\} = \frac{a}{s^2 - a^2}$$

$$4 - L\{coshax\} = \frac{s}{s^2 - a^2}$$

Example:

Find Laplace transforms for $L\{5e^{2x} - 3sin4x + x\}$.

Solution.

$$L\{5e^{2x} - 3sin4x + x\} = 5L\{e^{2x}\} - 3L\{sin4x\} + L\{x\}$$

$$= 5\left(\frac{1}{s-2}\right) - 3\left(\frac{4}{s^2+16}\right) + \frac{1}{s^2}$$

$$= \left(\frac{5}{s-2}\right) - \left(\frac{12}{s^2+16}\right) + \frac{1}{s^2}$$

Example:

Find Laplace transforms for $L\{e^{2x}x^2\}$.

Solution.

By above remark

$$L\{e^{ax}f(x)\} = F(s-a)$$

$$L\{x^2\} = \frac{2}{s^3}$$

$$L\{e^{3x}x^2\} = F(s-a) = \frac{2}{(s-3)^3}$$

Exercise:

1-Find Laplace transforms for $L\{e^{-2x}cos3x\}$.

2-Find Laplace transforms for $L\{e^{-7x}sinh5x\}$.

3-Find Laplace transforms for $L\{e^{-3x}x^3\}$.

4-Find Laplace transforms for $L\{e^{6x}cos\sqrt{2}x\}$.

Inverse Laplace transforms

solution of initial value problem by Laplace transforms, definitions of partial and Fourier series.

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